From Zero-freeness to Strong Spatial Mixing via a Christoffel-Darboux Type Identity

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Background

A 2-spin system is defined on a finite simple undirected graph $G = (V, E)$, it is associated with three parameters β, γ and λ. A partial configuration of this system refers to a mapping $\sigma : \Lambda \to \{\pm\}$ for some $\Lambda \subseteq V$ which may be empty. When $\Lambda = V$, it is a configuration and its weight denoted by $w(\sigma)$ is $\beta^{m_+(\sigma)}\gamma^{m_-(\sigma)}\lambda^{n_+(\sigma)}$, where $m_+(\sigma), m_-(\sigma)$ and $n_+(\sigma)$ denote respectively the number of $(+,+)$ edges, $(-,-)$ edges and vertices mapped to $+$. The *partition function* of a 2-spin system is defined to be

$$Z_G(\beta,\gamma,\lambda) := \sum_{\sigma:V\to\{+,-\}} w(\sigma).$$

We also define the partition function conditioning on a pre-described partial configuration σ_Λ (i.e., each vertex in Λ, called pinned vertex is pinned to be the spin $+$ or $-$) denoted by $Z_G^{\sigma_\Lambda}(\beta,\gamma,\lambda)$ to be $\sum_{\substack{\sigma:V\to\{+,-\}\\ \sigma|_\Lambda=\sigma_\Lambda}} w(\sigma)$ where $\sigma|_\Lambda$ denotes the restriction of the configuration σ on Λ.

1. FPTASes

Computing the partition function of the 2-spin system given an input graph G is a very basic counting problem, and it is known to be #P-hard for all complex valued parameters (β,γ,λ) but a few very restricted settings [1, 4, 5] (for example $\lambda = 0$ and $\beta\gamma = 1$). However, there exist two methods to derive fully polynomial-time deterministic approximation schemes (FPTAS).

The method associated with correlation decay, or more precisely *strong spatial mixing* (SSM) was originally developed by Weitz [14] for the hard-core model. This method compute Z_G via telescoping it into product of "marginal probabilities" $P_{G,v}^{\sigma_\Lambda} = \frac{Z_{G,v}^{\sigma_\Lambda,+}}{Z_G^{\sigma_\Lambda}}$ and applies the construction of Self-Avoiding Walk (SAW) tree which preserve the "probability" $P_{G,v}^{\sigma_\Lambda}$. Then, SSM property allows us to approximate partition functions.

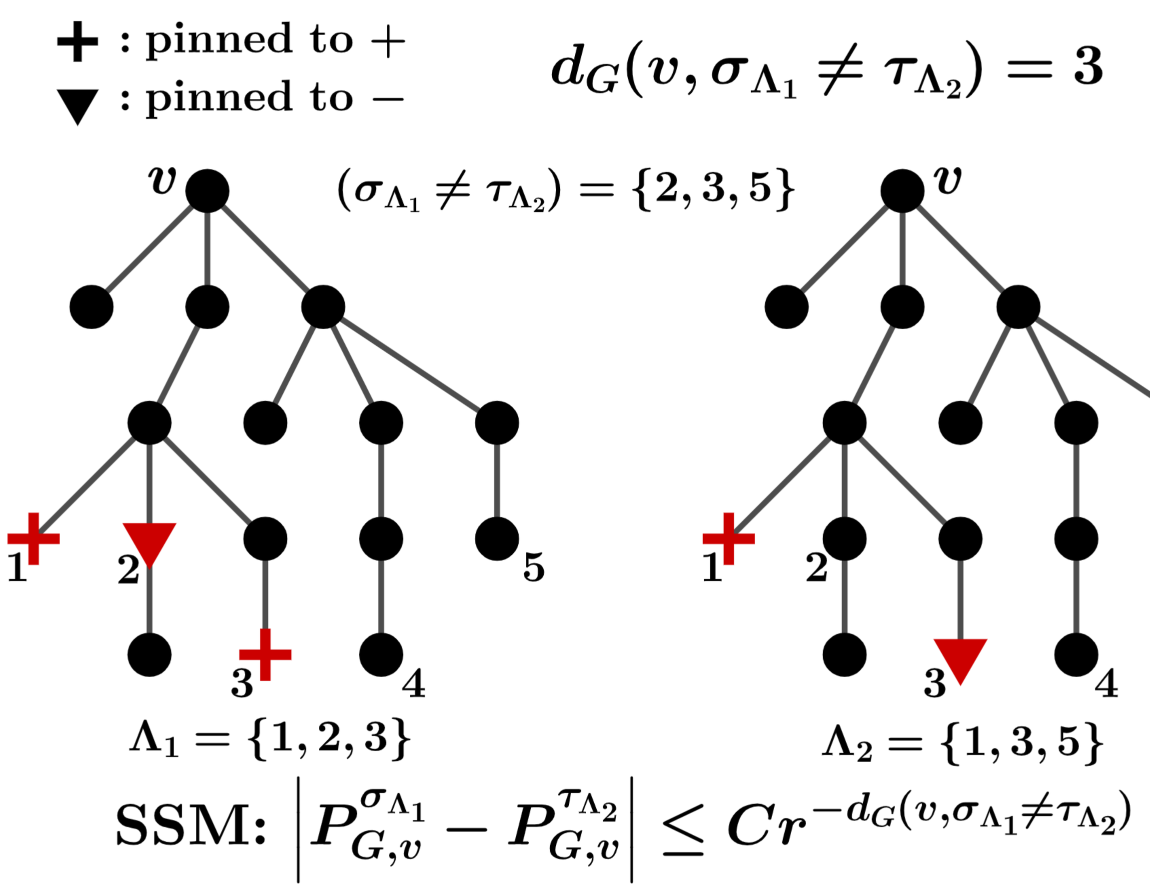

Figure 1: Strong Spatial Mixing

The method turning complex zero-free regions of the partition function into FPTASes was developed by Barvinok [2], and extended by Patel and Regts [8]. It is usually called the *Taylor polynomial interpolation* method. This method requires zero-freeness of partition functions in certain regions.

2. Zero-freeness Implies SSM

Regts [11] showed that zero-freeness in fact implies SSM for the hard-core model and other graph homomorphism models on all graphs of bounded degree. Our work generalizes this result to a general context. We prove that SSM can be obtained from the following kinds of zero-free regions.

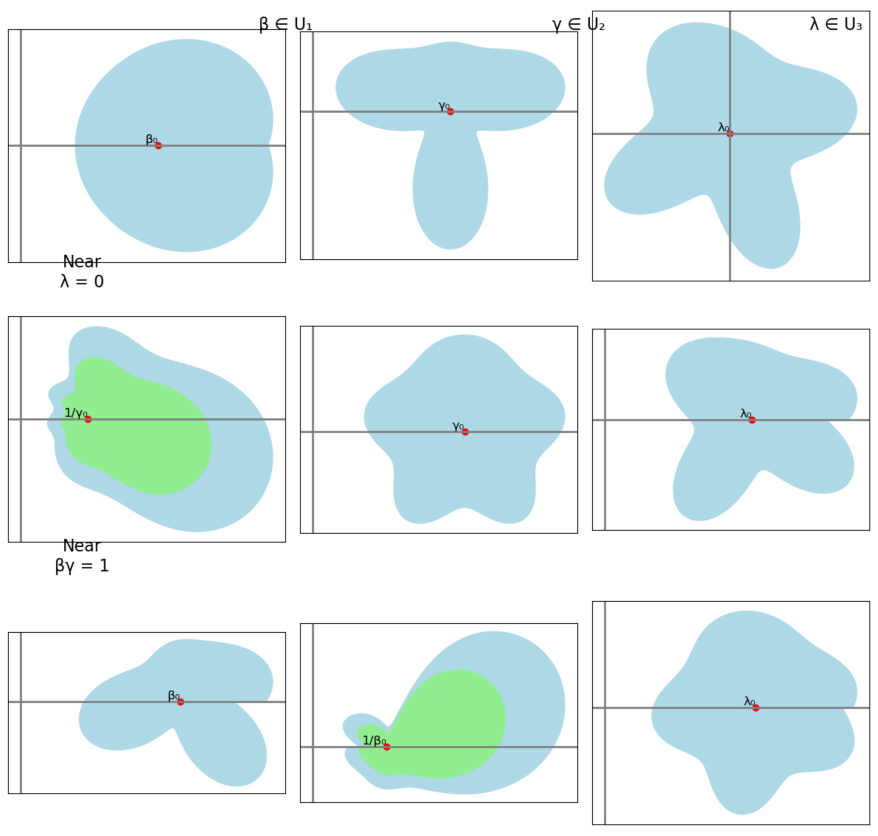

Figure 2: Regions in our result

The following illustrates our approach.

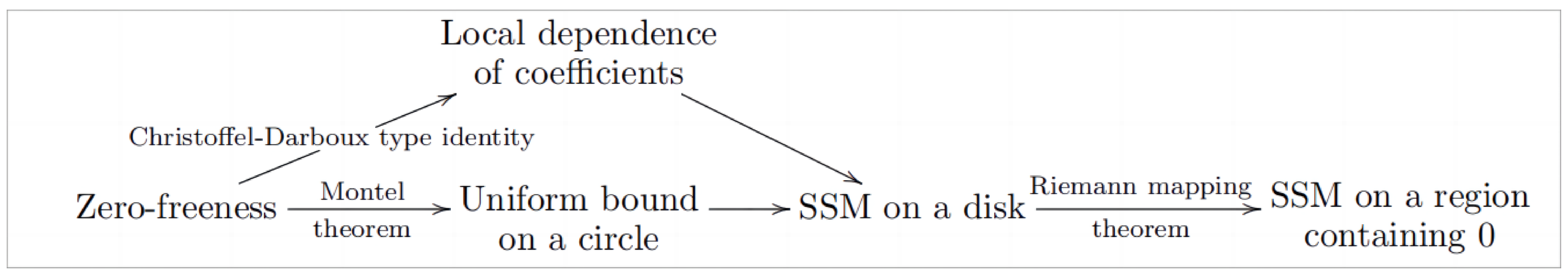

Figure 3: The structure of our approach

The local dependence of coefficients (LDC) property says that the coefficients of the first certain terms in the Taylor expansion of the rational function $P_{G,v}^{\sigma_{\Lambda_1}}(\lambda)$ near 0 depends only on structure of σ_{Λ_1} in a local neighborhood of v.

The uniform bound on a circle property says that there exists a circle $\partial\mathbb{D}_\rho$ around 0 on which the rational functions $P_{G,v}^{\sigma_\Lambda}(\lambda)$ have a uniform bound.

In Regts' work, the LDC property was proved by using Mayer's cluster expansion formula for polymer models to give an explicit expression for the Taylor series of the rational function $P_{G,v}^{\sigma_\Lambda}(\lambda)$ near 0, he applied this method to deal with the hard-core model and graph homomorphism model(Ising model without external field when restricted to 2-spin systems). However, a general 2-spin system is not known to be expressible as polymer models.

We prove LDC by establishing a Christoffel-Darboux type identity (which is also of its own interests) for the 2-spin system on trees and combining it with the SAW tree construction. This allows us to deal with 2-spin system of general coefficients.

Theorem 1 (Christoffel-Darboux type identity). *Let T be a tree, σ_Λ is a partial configuration, and $u \neq v$ are two vertices in $V\backslash\Lambda$ with distance $d(u,v)$ and p_{uv} the unique path in T connecting them. Denote the neighbors of points in p_{uv} by $v_1, \cdots, v_n$, and their components in $T \setminus p_{uv}$ by T_i. Then*

$$Z_{T,u,v}^{\sigma_\Lambda,+,+}(\beta,\gamma,\lambda)Z_{T,u,v}^{\sigma_\Lambda,-,-}(\beta,\gamma,\lambda) - Z_{T,u,v}^{\sigma_\Lambda,+,-}(\beta,\gamma,\lambda)Z_{T,u,v}^{\sigma_\Lambda,-,+}(\beta,\gamma,\lambda)$$
$$= \begin{cases} (\beta\gamma-1)^{d(u,v)}\lambda^{d(u,v)+1}\prod_{i=1}^{n}(\beta Z_{T_i,v_i}^{\sigma_\Lambda,+} + Z_{T_i,v_i}^{\sigma_\Lambda,-})(Z_{T_i,v_i}^{\sigma_\Lambda,+} + \gamma Z_{T_i,v_i}^{\sigma_\Lambda,-}) & , \textit{if } V(p_{uv}) \cap \Lambda = \emptyset \\ 0 & , \textit{if } V(p_{uv}) \cap \Lambda \neq \emptyset \end{cases}.$$

3. Applications

Our approach comprehensively turns all existing zero-free regions shown in the following table (to our best knowledge) of the partition function of the 2-spin system where pinned vertices are allowed into the SSM property.

	Model	Fixed parameters	Results
1	Hard-core	$\beta = 0, \gamma = 1$	$\{\lambda \in \mathbb{C} \mid \lvert\lambda\rvert < \frac{(d-1)^{d-1}}{d^d}\}$ [9]
2	Hard-core	$\beta = 0, \gamma = 1$	$\lambda \in \mathcal{N}(I), I = [0, \frac{(d-1)^{d-1}}{(d-2)^d})$ [9]
3	Hard-core	$\beta = 0, \gamma = 1$	$\lambda \in \mathcal{N}(0)$ irregular shape [3]
4	Ising	$\lambda = 1$	$\beta \in \mathcal{N}(I), I = (\frac{d-2}{d}, \frac{d}{d-2})$ [7]
5	Ising	$\beta \in (\frac{d-2}{d}, 1)$	$\{\lambda \in \mathbb{C} \mid \lvert\arg(\lambda)\rvert < \theta(\beta)\}$ [10]
6	2-Spin	None	Four regions [12]

Moreover, we extend our approach to handle the 2-spin system with non-uniform external fields. As an application, we obtain a new SSM result for the non-uniform ferromagnetic Ising model from the celebrated Lee-Yang circle theorem [6].

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