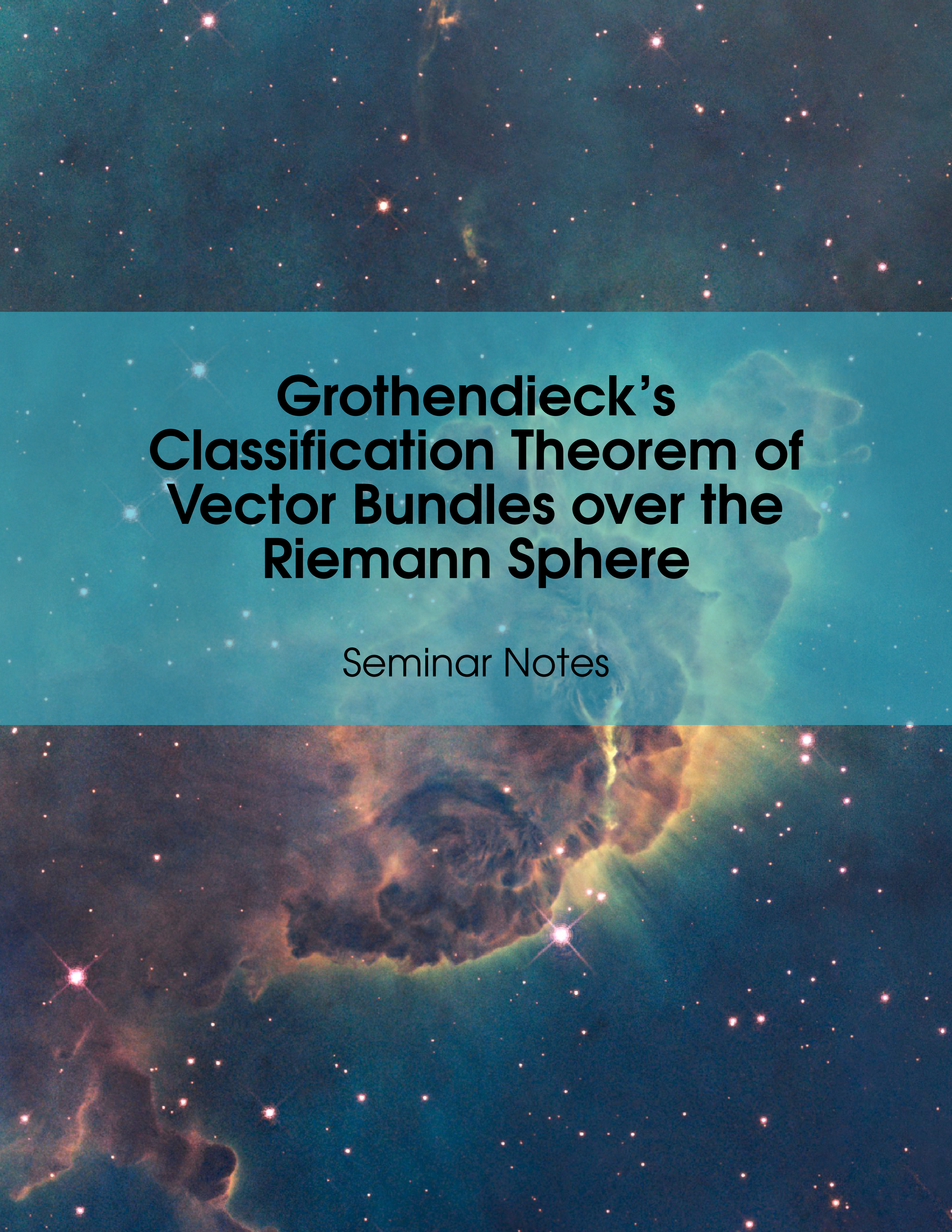
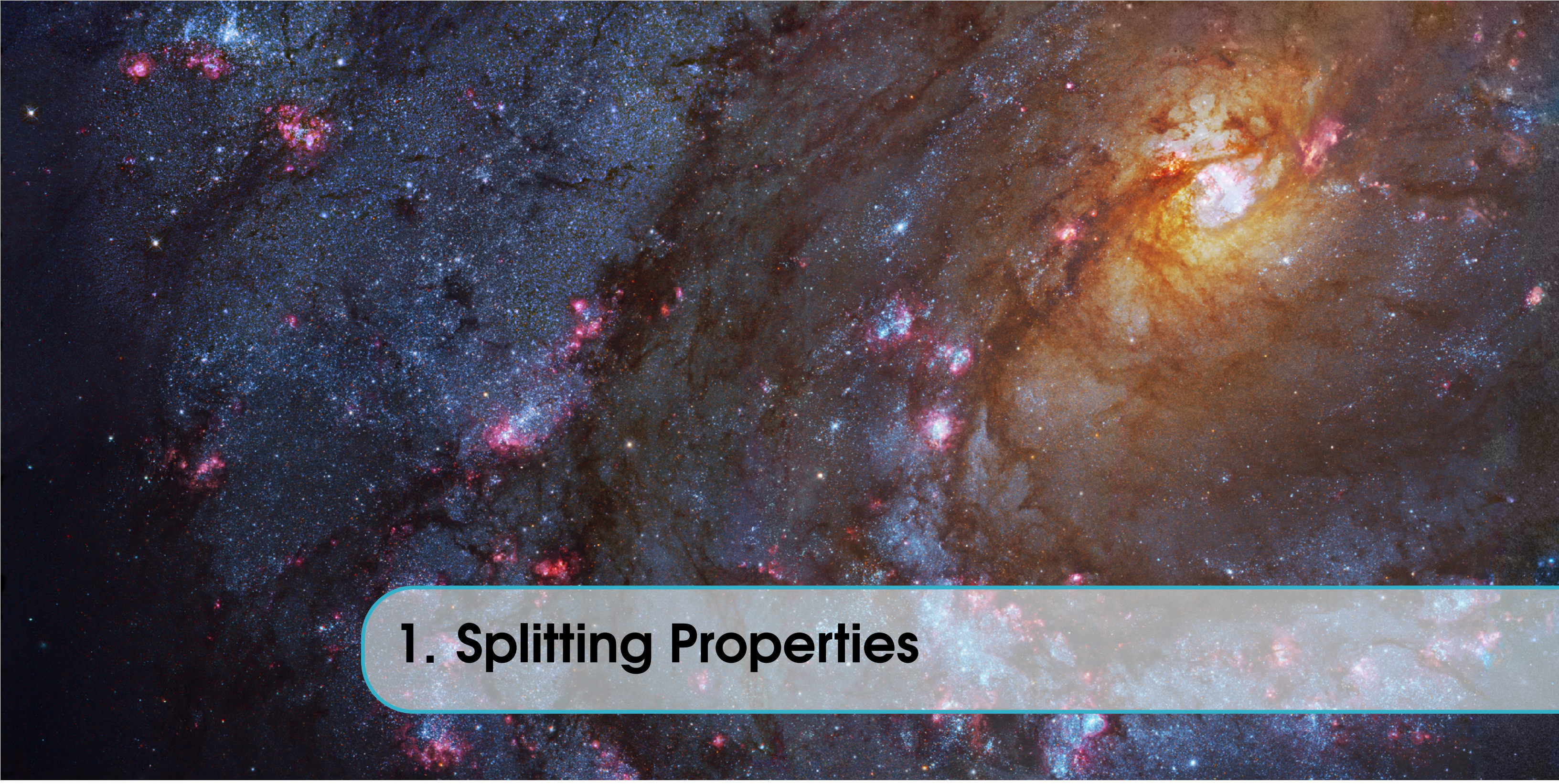


Grothendieck's Classification Theorem of Vector Bundles over the Riemann Sphere

Seminar Notes

Contents



1. Splitting Properties

1.1 Notations and Facts

Let X be a compact Riemann surface, E be a holomorphic vector bundle over X and $F \subset E$ be a holomorphic subbundle. Recall that by definition F is a submanifold of E.

One can define the quotient bundle:

Theorem 1.1.1 There exists a unique holomorphic vector bundle structure on

$$E/F := \bigsqcup_{x \in X} (E_x/F_x) \to X$$

which satisfies the following property: each homomorphism between holomorphic vector bundles $f : E \to G$ which vanishes on F induces a homomorphism between holomorphic vector bundles $\overline{f} : E/F \to G$.

Then we have a short exact sequence:

$$0 \to F \xrightarrow{i} E \xrightarrow{p} G := E/F \to 0$$

Tensoring with the dual bundle G^*, we obtain another short exact sequence

$$0 \to Hom(G,F) \xrightarrow{i^*} Hom(G,E) \xrightarrow{p^*} End(G) \to 0$$

since the tensor functor for the category of vector bundles is exact.

This short exact sequence of vector bundles induces a long exact sequnce of corrsponding cohomology groups

$$0 \to Hom_X(G,F) \to Hom_X(G,E) \to End_X(G) \to H^1(Hom(G,F)) \to \cdots$$

Now we turn to prove some properties that will be used later.

1.2 Statements and proofs

Theorem 1.2.1 $E \simeq F \oplus G$ if and only if there exists a homomorphism of vector bundles $f : G \to E$ such that the composition $G \xrightarrow{f} E \to G$ is the identity map

Proof. If $E \simeq F \oplus G$, then a such homomorphism abviously exists.
Conversely, we consider the map

$$T : E \to F \oplus G, (x,e) \mapsto (x, i^{-1}(e - f(p(e))) \oplus g(e))$$

we shall verify that it is a map between vector bundles (trivial), it is holomorphic (since F is a submanifold and i is the nature imbedding) and it is bijective (trivial). ■

Theorem 1.2.2 If $H^1(Hom(G,F)) = 0$, then $E \simeq F \oplus G$.

Proof. By exactness, $Hom_X(G,E) \to End_X(G)$ is surjective. In particular, there exists a homomorphism of vector bundles $f : G \to E$ such that the composition $G \xrightarrow{f} E \to G$ is the identity map, and the previous theorem applies. ■

2. Riemann-Roch for Vector Bundles

2.1 Case of Line Bundle

Recall that we have a bijective correspondence between isomorphic classes of holomorphic line bundle and equivalent classes of divisor, under this correspondence, the sheaf of germs of holomorphic sections of a holomorphic line bundle L can be identified with $\mathscr{O}_D$ for the corresponding divisor D.

Note $h^i(L)$ the dimension of the i-th Čech cohomology group associated to the sheaf of germs of holomorphic sections of L. This notation will also be used later for vector bundles of higher rank, by finiteness theorem $h^i < \infty, i = 0, 1$. By the known version of Riemann-Roch theorem, we obtain the Riemann-Roch theorem for line bundle:

Theorem 2.1.1 — Riemann-Roch for line bundles. For a holomorphic line bundle L, we have

$$h^0(L) - h^1(L) = \deg L - 1 + g$$

where g is the genus of the given Riemann surface and $\deg L$ is the degree of L, defined as the degree of the corresponding divisor.

2.2 General Case

In this section, we shall generalise the Riemann-Roch theorem to holomorphic vector bundles of higher rank.

Definition 2.2.1 For a holomorphic vector bundle E of rank r, we define its determinant line bundle

$$\det(E) := \wedge^r E$$

and its degree $\deg E := \deg \det(E)$.

We easily see that

$$\begin{cases} \det(A\oplus B) = \det(A)\otimes\det(B) \\ \det(A\otimes B) = \bigotimes^{r(B)}\det(A)\bigotimes^{r(A)}\det(B). \end{cases}$$

Moreover, for the exact sequence in the previous section, we have

$$\deg E = \deg F + \deg G$$

the proof is simple if we write down the transition map as up-triangularly blocked matrix.

Theorem 2.2.1 Every holomorphic vector bundle of rank>1 contains a line bundle as subbundle.

Proof. By finiteness theorem, one can proof that every holomorphic vector bundle E of positive rank has a global meromorphic section s which does not vanish identically. (c.f.GTM81,29.17)

For the rest, see R.C.Gunning2, p61,Lemma11:

Lemma 2.2.2 Let Ψ be a holomorphic vector bundle of rank $m>1$ over a Riemann surface M and F a non-trivial meromorphic section of Ψ. Then Ψ has a line subbundle ψ with $\deg\psi = \deg(F)$.

Proof. Let (U_α) be a covering of local trivialization, $\Psi_{\alpha\beta}$ be the corresponding transition matrix and (F_α) represent the section F. We have

$$F_\alpha(p) = \Psi_{\alpha\beta}(p)F_\beta(p), \forall p\in U_\alpha\cap U_\beta.$$

By refining the covering, we can suppose that F_α is holomorphic and non-singular (not all component vanish) in U_α except at (at most) one point.

By refining again, suppose all U_α are coordinate neighborhoods with coordinate z_α and the exceptional point is the origin $z_\alpha = 0$. Then there exists $r_\alpha\in\mathbb{Z}$ s.t. $z_\alpha^{r_\alpha}F_\alpha(z_\alpha)$ is holomorphic and non-singular on U_α.

By refining again if needed, there is a holomorphic non-singular matrix valued function Ψ_α s.t.

$$\Psi_\alpha z_\alpha^{r_\alpha}F_\alpha = e_1,$$

where

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Now we consider the equivalent transition matrix $\Psi'_{\alpha\beta} := \Psi_\alpha\Psi_{\alpha\beta}\Psi_\beta^{-1}$, under these transition map, the section F is expressed as

$$F'_\alpha = \Psi_\alpha F_\alpha = z_\alpha^{-r_\alpha}e_1$$

Then, in $U_\alpha\cap U_\beta$, we have

$$z_\alpha^{-r^\alpha}e_1 = \Psi'_{\alpha\beta}z_\beta z_\beta^{-r_\beta}e_1,$$

thus the matrix $\Psi'_{\alpha\beta}$ has the form

$$\begin{pmatrix} \psi_\alpha & \star & \cdots & \star \\ 0 & \star & \cdots & \star \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \star & \cdots & \star \end{pmatrix}$$

then ψ_α defined a line subbundle ψ with section F, hence $\deg \psi = \deg(F)$. ■

■

Theorem 2.2.3 — Riemann-Roch for vector bundles. Let E be holomorphic vector bundle of rank r, then

$$h^0(E) - h^1(E) = \deg E - r(g-1).$$

Proof. By induction on r. It suffices to show that for the exact sequence in the previous section with F a line bundle, we have

$$h^0(E) - h^1(E) = (h^0(F) - h^1(F)) + (h^0(G) - h^1(G))$$

by inducing long exact sequence, it suffices to show that $h^2(L) = 0$ for any line bundle L. See R.C.Gunning1, p74, Theorem8.

(Fine sheaf, fine resolution, Dolbeault's theorem for fine resolution. c.f. R.C.Gunning1, p37, Theorem3, or Section 4.5 of book of MEI Jiaqiang) ■

3. Grothendieck's Theorem

3.1 Case of Rank 2

From now on, we suppose that $X = \mathbb{P}^1$. First, recall a vanishing theorem:

Theorem 3.1.1 L is a holomorphic line bundle, then

$$\deg L \leq -1 \implies h^0(L) = 0; \qquad \deg L \geq -1 \implies h^1(L) = 0.$$

Now we prove the Grothendieck's theorem for rank 2 holomorphic vector bundles:

Lemma 3.1.2 Let E be a rank 2 holomorphic vector bundle, then E is isomorphic to a direct sum of line bundles.

Proof. By tensoring a line bundle, we can suppose without lost of generality that $\deg E = 0$ or $\deg E = -1$. Then it follows from the Riemann-Roch theorem that $h^0(E) \neq 0$, which implies that there is a line subbundle of non-negative degree (correspondent to a non-trivial holomorphic section).

We take the exact sequence as in the first section and we can verify that Theorem1.2.2 applies by using the previous vanishing theorem. ■

3.2 General Case

Theorem 3.2.1 Every holomorphic vector bundle over $\mathbb{P}^1$ splits to direct sum of line bundles, the decomposition is unique up to permutation and isomorphism.

Proof. We first prove the existence of decomposition by induction.

By Riemann-Roch we can show that degree of line subbundle of vector bundle E is up-bounded by $h^0(E) - 1$. (For line subbundle $L \subset E$, since $H^0(L) \subset H^0(E)$, we have $\deg L + 1 = h^0(L) - h^1(L) \leq h^0(E)$.) Thus, we can take a line subbundle of E with maximal degree, and we consider the exact sequence

$$0 \to L \to E \to E/L \simeq \bigoplus_{i=1}^{r-1} L_i \to 0$$

by hypothesis of induction. We claim that $\deg L_i \leq \deg L$ and hence $h^1(L_i^* \otimes L) = 0$ and the splitting property applies.

Now we turn to prove the claim: consider the exact sequence

$$0 \to L \to \tilde{L_i} \to L_i \to 0$$

where $\tilde{L_i}$ is the preimage of L under the projection map. Apply the conclusion for the case of rank 2, $\tilde{L_i}$ contains a line subbundle of degree

$$\geq \frac{\deg \tilde{L_i}}{2},$$

since L is a line subbundle of E with maximal degree, we have

$$\deg L \geq \frac{\deg \tilde{L_i}}{2}$$

thus $\deg L_i \leq \deg L$.

By tensoring the dual bundle and comparing h^0 we can prove that the line bundle of highest degree in two decompositions must coincide. Repeating this argument, we can show the uniqueness of decomposition. ∎