Abel's Theorem and Compact Riemann Surfaces of Genus One

Seminar Notes

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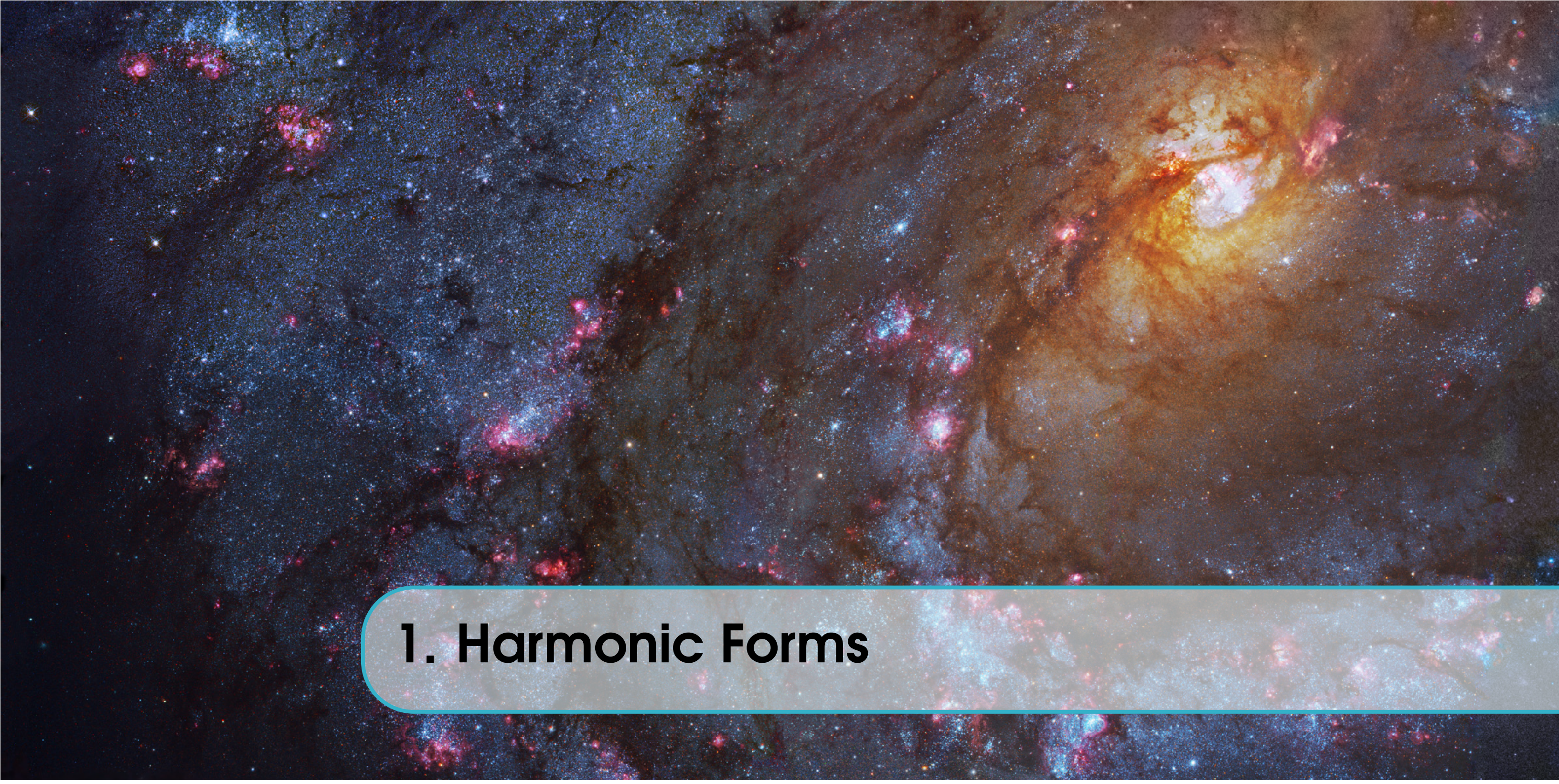

1. Harmonic Forms

1.1 Complex Conjugation and the $\star$-operator

Let X be a Riemann surface.

Given an 1-form $\omega \in \mathscr{E}^{(1)}(X)$, we can, locally, write it as

$$\omega = \sum_j f_j \mathrm{d} g_j, f_j, g_j \in \mathscr{E}(X),$$

and then we define its complex conjugation as:

Definition 1.1.1 — Complex Conjugation.

$$\bar{\omega} := \sum_j \bar{f}_j \mathrm{d} \bar{g}_j$$

It is easy to see that this definition is independent of choice of local representation of ω.

Definition 1.1.2 We give the following related definitions:

A differentiable $1-$form $\omega \in \mathscr{E}^{(1)}(X)$ is called real if $\omega = \bar{\omega}$.

The real part of $\omega \in \mathscr{E}^{(1)}(X)$ is

$$\mathrm{Re}(\omega) := \frac{\omega + \bar{\omega}}{2}.$$

An 1-form is said to be anti-holomorphic if it is the complex conjugation of some holomorphic 1-form. The space of all anti-holomorphic 1-forms is noted $\bar{\Omega}(X)$.

We also know that ω can be uniquely decomposed as

$$\omega = \omega_1 + \omega_2, \omega_1 \in \mathscr{E}^{1,0}(X), \omega_2 \in \mathscr{E}^{0,1}(X).$$

With this decomposition, We introduce the $\star$-operator:

Definition 1.1.3 — The $\star$-operator.

$$\star\omega := i(\bar{\omega}_1 - \bar{\omega}_2).$$

Fact 1.1.1 Let $\omega \in \mathscr{E}^{(1)}(X), \omega_1 \in \mathscr{E}^{1,0}(X)\omega_2 \in \mathscr{E}^{0,1}(X)$ and $f \in \mathscr{E}(X)$, then we have:

1. The $\star$-operator is an $\mathbb{R}$-linear isomorphism of $\mathscr{E}^{(1)}(X)$ mapping $\mathscr{E}^{0,1}(X)$ to $\mathscr{E}^{1,0}(X)$ and vice versa.
2. $\star\star\omega = -\omega, \star\bar{\omega} = \star\bar{\omega}$,
3. $\mathrm{d}\star(\omega_1 + \omega_2) = i\mathrm{d}'\bar{\omega}_1 - i\mathrm{d}''\bar{\omega}_2$,
4. $\star\mathrm{d}'f = i\mathrm{d}''\bar{f}, \star\mathrm{d}''f = i\mathrm{d}'\bar{f}$,
5. $\mathrm{d}\star\mathrm{d}f = 2i\mathrm{d}'\mathrm{d}''\bar{f}$.

1.2 The deRham-Hodge Theorem

Definition 1.2.1 — Harmonic Forms. The 1-form $\omega \in \mathscr{E}^{(1)}(X)$ is called harmonic if

$$\mathrm{d}\omega = \mathrm{d}\star\omega = 0.$$

The vector space of all harmonic 1-form is noted $\mathrm{Harm}^1(X)$.

By the above facts we can easily prove the following theorem:

Theorem 1.2.1 For $\omega \in \mathscr{E}^{(1)}(X)$, the following conditions are equivalent:

(1) ω is harmonic,
(2) $\mathrm{d}'\omega = \mathrm{d}''\omega = 0$,
(3) $\omega = \omega_1 + \omega_2$ where $\omega_1 \in \Omega(X)$ and $\omega_2 \in \bar{\Omega}(X)$,
(4) For all $a \in X$ there exists an open neighborhood U of a and a harmonic function f on U such that $\omega = \mathrm{d}f$.

By this theorem we can write

$$\mathrm{Harm}^1(X) = \Omega(X) \oplus \bar{\Omega}(X),$$

thus if X is compact of genus g, then

$$\dim \mathrm{Harm}^1(X) = 2g.$$

From now on we suppose that X is compact.

Definition 1.2.2 — Scalar Product in $\mathscr{E}^{(1)}(X)$. For $\omega_1, \omega_2 \in \mathscr{E}^{(1)}(X)$,

$$< \omega_1, \omega_2 > := \iint_X \omega_1 \wedge \star\omega_2.$$

It is easy to verify that this scalar product is well-defined.

Fact 1.2.2 The four spaces $\mathrm{d}'\mathscr{E}(X), \mathrm{d}''\mathscr{E}(X), \Omega(X)$ and $\bar{\Omega}(X)$ are pairwise orthogonal.

R Hint for proof: use the fact

$$\iint_X \mathrm{d}\omega = 0.$$

Fact 1.2.3 The two spaces $\mathrm{d}\mathscr{E}(X)$ and $\star\mathrm{d}\mathscr{E}(X)$ are orthogonal and

$$\mathrm{d}\mathscr{E}(X)\oplus\star\mathrm{d}\mathscr{E}(X)=\mathrm{d}'\mathscr{E}(X)\oplus\mathrm{d}''\mathscr{E}(X).$$

R Hint for proof: recall the fact 1.1.1.4.

The above discussion gives the following facts:

Theorem 1.2.4 An exact and harmonic 1-form on a compact Riemann surface vanishes and all harmonic functions on a compact Riemann surface is constant.

and then useful fact follows:

Fact 1.2.5 Suppose $\sigma\in\mathrm{Harm}^1(X)$ and $\omega\in\Omega(X)$, then:

$\sigma=0$ if and only if for every closed curve γ on X one has

$$\int_\gamma \sigma=0;$$

$\omega=0$ if and only if for every closed curve γ on X one has

$$\int_\gamma \mathrm{Re}(\omega)=0.$$

By combining Dolbeault's Theorem and Fact 1.2.2 and by comparing dimensions, we obtain that:

Theorem 1.2.6 There is an orthogonal decomposition

$$\mathscr{E}^{0,1}(X)=\mathrm{d}''\mathscr{E}(X)\oplus\bar{\Omega}(X).$$

This theorem implies directly a method to justify the existence of solution for the equation $\mathrm{d}''f=\sigma$ where $\sigma\in\mathscr{E}^{0,1}(X)$ is given:

Theorem 1.2.7 Suppose that $\sigma\in\mathscr{E}^{0,1}(X)$, then the equation $\mathrm{d}''f=\sigma$ has a solution $f\in\mathscr{E}(X)$ if and only if for all $\omega\in\Omega(X)$, one has

$$\iint_X \sigma\wedge\omega=0.$$

Taking complex conjugates in Theorem 1.2.6 and then applying Facts 1.2.2 and 1.2.3, we have:

Theorem 1.2.8 There is an orthogonal decomposition

$$\mathscr{E}^{(1)}(X)=\star\mathrm{d}\mathscr{E}(X)\oplus\mathrm{d}\mathscr{E}(X)\oplus\mathrm{Harm}^1(X).$$

One can also prove that

Theorem 1.2.9

$$\ker\left(\mathscr{E}^{(1)}(X)\xrightarrow{d}\mathscr{E}^{(2)}(X)\right)=\mathrm{d}\mathscr{E}(X)\oplus\mathrm{Harm}^1(X).$$

Combine this theorem with deRham's Theorem, we obtain finally the deRham-Hodge Theorem:

Theorem 1.2.10 For a compact Riemann surface X, we have

$$H^1(X,\mathbb{C}) \simeq \mathrm{Rh}^1(X) \simeq \mathrm{Harm}^1(X).$$

R it allows us to compute the first Betti number of X:

$$b_1(X) := \dim H^1(X,\mathbb{C}) = 2g,$$

where g is the genus of X.

1.3 The "Main Theorem"

Following the above discussion, we state and prove the following theorem (the "Main Theorem" in Donaldson's book *Riemann Surfaces*):

Theorem 1.3.1 Let X be a compact Riemann surface and $\omega \in \mathscr{E}^{(2)}(X)$, then the equation

$$\mathrm{d}'\mathrm{d}'' f = \omega$$

has a solution $f \in \mathscr{E}(X)$ if and only if

$$\iint\limits_X \omega = 0.$$

Proof. From Theorem 1.2.6 we obtain that

$$\mathrm{d}\mathscr{E}^{0,1}(X) = \mathrm{d}'\mathrm{d}''\mathscr{E}(X).$$

hence by Dolbeault's Theorem and Serre's Duality Theorem we have

$$\mathscr{E}^{(2)}(X)/\mathrm{d}'\mathrm{d}''\mathscr{E}(X) \simeq \mathscr{E}^{(2)}(X)/\mathrm{d}\mathscr{E}^{0,1}(X) \simeq \mathscr{E}^{(2)}(X)/\mathrm{d}\mathscr{E}^{1,0}(X) \simeq H^1(X,\Omega) \simeq H^0(X,\mathscr{O}) \simeq \mathbb{C}.$$

We view the integration as a linear form on $\mathscr{E}^{(2)}(X)/\mathrm{d}'\mathrm{d}''\mathscr{E}(X)$ which is (well defined and) non-zero, hence an isomorphism, and the result follows. ■

2. Abel's Theorem: Incomplete Version

We always denote X a compact Riemann surface, g its genus and $D \in \mathrm{Div}(X)$ a divisor on X.

2.1 Solution and Weak Solution of a Divisor

Definition 2.1.1 — Solution. A solution of D is a meromorphic function $f \in \mathscr{M}(X)$ such that $(f) = D$.

> **R** If D has a solution, then $\deg D = 0$ by Residue Theorem.

In another word, a solution is a meromorphic function with asymptotic behavior at certain points described by the divisor.

It is not easy for a divisor to have a solution, but we will prove that if $\deg D = 0$, one can always find a function with asymptotic behavior described by D, such a function is called a weak solution, as we are going to define:

Definition 2.1.2 — Weak Solution. Note

$$X_D := \{x \in X : D(x) \geq 0\}.$$

A weak solution of D is a smooth function f on X_D such that for all $a \in X$ there exists a coordinate neighborhood (U, z) with $z(a) = 0$ and a function $\psi \in \mathscr{U}$ with $\psi(a) \neq 0$, such that

$$f = \psi z^k \text{ on } U \cap X_D, \text{ where } k = D(a).$$

> **R** If f_1 (resp. f_2) is a weak solution of D_1 (resp. D_2), then $f_1 f_2$ is a weak solution of $D_1 + D_2$.

Lemma 2.1.1 Suppose $a_1, \cdots, a_n \in X$ distinct and $k_1, \cdots, k_n \in \mathbb{Z}$. Suppose D is the divisor on X with $D(a_j) = k_j, j = 1, \cdots, n$ and $D(a) = 0$ otherwise. Let f be a weak solution of D. Then for all

$g \in \mathscr{E}(X)$, we have

$$\frac{1}{2\pi i}\iint_X \frac{\mathrm{d}f}{f}\wedge \mathrm{d}g = \sum_{j=1}^{n} k_j g(a_j).$$

R Hint for proof: construct bump finctions on neighborhoods of a_j and use Stokes' Theorem.

2.2 Chains and Cycles

Definition 2.2.1 — 1-chain. We define the 1-chain group $C_1(X)$ as the free abelian group generated by all curves on X. Elements of this group are called 1-chains.

For an element $c \in C_1(X)$, we can write

$$c = \sum_{j=1}^{k} n_j c_j,$$

where $n_j \in \mathbb{Z}$ and c_j are curves.
One can naturally define integral of a closed 1-form over a 1-chain:

Definition 2.2.2 — Integral. For $c \in C_1(X)$ and a closed form $\omega \in \mathscr{E}^{(1)}(X)$,

$$\int_c \omega := \sum_{j=1}^{k} n_j \int_{c_j} \omega.$$

To introduce the concept of 1-cycles, we should define a boundary operator:

Definition 2.2.3 — Boundary Operator. For a curve c on X, set $\partial c = 0$ be the zero divisor if $c(0) = c(1)$, otherwise let ∂c be the divisor with $+1$ at $c(1)$ and -1 at $c(0)$ and zero at all other points. This definition can be extened to $C_1(X)$, hence induces a homomorphism of group

$$\partial : C_1(X) \to \mathrm{Div}(X).$$

Observation 2.2.1 $\mathrm{Im}\partial = \{D \in \mathrm{Div}(X) : \deg D = 0\}$.

Definition 2.2.4 — 1-Cycles. The 1-cycle group of X is defined as $Z_1(X) := \ker \partial$.

Theorem 2.2.2 For an 1-chain $c \in \mathrm{Div}(X)$, there exists a weak solution of ∂c such that, for all closed form $\omega \in \mathscr{E}^{(1)}(X)$ one has

$$\int_c \omega = \frac{1}{2\pi i}\iint_X \frac{\mathrm{d}f}{f}\wedge \omega.$$

R Hint for proof: since $[0,1]$ is compact, using the remark after Definition 2.1.2, it suffices to consider the case where c is a curve and $c([0,1])$ is contained in a coordinate neighborhood biholomorphic to the unit disk. In this case, construct directly the weak solution f with help of a bump function.

2.3 Abel's Theorem: Incomplete Version

Theorem 2.3.1 If there exists a 1-chain $c \in C_1(X)$ with $\partial c = D$ such that for all $\omega \in \Omega(X)$ we have

$$\int_c \omega = 0,$$

then D has a solution.

Proof. Take a weak solution given by Theorem 2.2.2, we have for all $\omega \in \Omega(X)$,

$$0 = \int_c \omega = \frac{1}{2\pi i} \iint_X \frac{\mathrm{d}f}{f} \wedge \omega = \frac{1}{2\pi i} \iint_X \frac{\mathrm{d}'' f}{f} \wedge \omega.$$

Hence by Theorem 1.2.7, there exists a function $g \in \mathscr{E}(X)$ such that

$$\mathrm{d}'' g = \frac{\mathrm{d}'' f}{f} \wedge \omega.$$

Then we verify that $F := e^{-g} f$ is a solution of D. ■

R In fact, if D has a solution, one can conclude that there exists a 1-chain $c \in C_1(X)$ with $\partial c = D$ such that for all $\omega \in \Omega(X)$ we have

$$\int_c \omega = 0.$$

We will prove it later.

3. Riemann Surfaces of Genus One

3.1 Classification of Tori

For two tori $\mathbb{C}/\Gamma_1$ and $\mathbb{C}/\Gamma_2$, suppose that

$$\Gamma_j = a_j\mathbb{Z} \oplus b_j\mathbb{Z}, j = 1,2.$$

Let

$$\gamma_j := \frac{a_j}{b_j}, j = 1,2.$$

We have already obtained the following classification result:

Theorem 3.1.1 $\mathbb{C}/\Gamma_1 \simeq \mathbb{C}/\Gamma_2$ if and only if there exists a matrix

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \in GL_2(\mathbb{Z})$$

such that

$$\gamma_1 = \frac{g_{11}\gamma_2 + g_{12}}{g_{21}\gamma_2 + g_{22}}$$

3.2 Period Lattices and Jacobi Variety

For a compact Riemann surface of genus $g \geq 1$, take a basis $\omega_1, \cdots, \omega_g$ of $\Omega(X)$. We define the period lattice of X with respect to this basis in the following way:

Definition 3.2.1 — Period Lattices.

$$\mathrm{Per}(\omega_1, \cdots, \omega_g) := \left\{ \left(\int_\alpha \omega_1, \cdots, \int_\alpha \omega_g \right) : \alpha \in \pi_1(X) \right\}.$$

We shall show that the period lattice is a lattice in $\mathbb{C}^g$.

Lemma 3.2.1 There exists g distinct points $a_1, \cdots, a_g \in X$ such that every holomorphic 1-form vanishing at all a_j is identically zero.

Proof. For $a \in X$, notice that

$$H_a := \{\omega \in \Omega(X) : \omega(a) = 0\}$$

is a subspace of $\Omega(X)$ with codimension 0 or 1, since the intersection of all H_a is zero and

$$\dim \Omega(X) = g,$$

the result follows. ∎

Theorem 3.2.2 $\Gamma := \mathrm{Per}(\omega_1, \cdots, \omega_g)$ is a lattice in $\mathbb{C}^g$.

In the proof, we will admit and use the following theorem:

Theorem 3.2.3 A subgroup $\Gamma \subset \mathbb{C}^N$ is a lattice precisely if both of the following conditions hold:
(1) Γ is discrete.
(2) Γ can $\mathbb{R}$-generate $\mathbb{C}^N$.

Proof of Theorem 3.2.2. Chose $a_1, \cdots, a_g$ as in lemma 3.2.1 and take disjoint simply connected coordinate neighborhoods (U_j, z_j) of a_j with $z_j(a_j) = 0$ and

$$\omega_i = \phi_{ij} \mathrm{d}z_j \text{ on } U_j.$$

First, show that Γ is discrete: by lemma 3.2.1, the matrix

$$A := (\phi_{i,j}(a_j))_{1 \leq i,j \leq g}$$

has rank g. Now we define a mapping

$$F : U_1 \times \cdots \times U_g \to \mathbb{C}^g$$

as follows:

$$F(x_1, \cdots, x_g) := \left(\sum_{j=1}^{g} \int_{a_j}^{x_j} \omega_1, \cdots, \sum_{j=1}^{g} \int_{a_j}^{x_j} \omega_g \right).$$

Clearly F is holomorphic and its Jacobian at $a = (a_1, \cdots, a_g)$ is the invertible matrix A. Hence we can suppose that F is a diffeomorphism and

$$W := F(U_1 \times \cdots \times U_g)$$

is a neighborhood of $F(a) = 0$. It suffices to show that $\Gamma \cap W = \{0\}$.
Suppose to the contrary that there exists a point

$$x = (x_1, \cdots, x_g) \in U_1 \times \cdots \times U_g, x \neq a$$

such that $F(x) \in \Gamma$. By Theorem 2.3.1 one can take a solution f of the divisor

$$x_1 + \cdots + x_g - a_1 - \cdots - a_g$$

Let c_j be the residue of f at a_j, since $a \neq x$, there is some j such that $c_j \neq 0$. By Residue Theorem,

$$0 = \mathrm{Res}(f\omega_i) = \sum_{j=1}^{k} c_j \phi_{i,j}(a_j), i = 1, \cdots, g.$$

This is impossible since $(\phi_{i,j}(a_j))_{1 \leq i,j \leq g}$ has rank g.

Then, we show that Γ can $\mathbb{R}$-generate $\mathbb{C}^g$: otherwise, we could find a non-trivial $\mathbb{R}$-linear form on $\mathbb{C}^g$ vanishing on Γ, represent this real form as the real part of some complex linear form, we get a vector $(c_1, \cdots, c_g) \in \mathbb{C}^g \setminus \{0\}$ such that for all $\alpha \in \pi_1(X)$,

$$\mathrm{Re}\left(\sum_{j=1}^{g} c_j \int_{\alpha} \omega_j\right) = 0.$$

By Fact 1.2.5 we conclude that $c_1\omega_1 + \cdots + c_g\omega_g = 0$,
a contradiction! ■

Hence, we can introduce the concept of the Jacobi variety:

Definition 3.2.2 — Jacobi Variety. The Jacobi variety of X is the compact complex manifold

$$\mathrm{Jac}(X) := \mathbb{C}^g/\mathrm{Per}(\omega_1, \cdots, \omega_g).$$

R This definition is independent of the choice of the basis $\omega_1, \cdots, \omega_g$ up to a biholomorphism.

3.3 Discussion of Case of Genus One

Now we suppose that $g = 1$. Take a point $a \in X$, a basis ω of $\Omega(X)$ and its corresponding period lattice $\Gamma := \mathrm{Per}(\omega)$. We define the following map $J : X \to \mathrm{Jac}(X)$:

$$J(x) := \int_a^x \omega \mod \Gamma.$$

Theorem 3.3.1 J is a biholomorphism.

Proof. It is clear that J is well defined and holomorphic, it is also non-constant since ω is non-trivial.

By the open map theorem for holomorphic map between Riemann surfaces, we deduce that J is surjective.

We show that J is also injective: otherwise, by Theorem 2.3.1 one has a meromorphic function with a single pole of order one. It thus induces a 1-sheeted holomorphic covering (hence a biholomorphism) from X to $\mathbb{P}^1$, a contradiction! ■

R Now we can finish the classification of compact Riemann surfaces of genus 1 since each such Riemann surface is biholomorphic to a torus.

4. Abel's Theorem: Complete Version

Still, let X be a compact Riemann surface of genus 1 and D be a divisor with $\deg D = 0$ on X.

Theorem 4.0.1 D has a solution if and only if there exists a 1-chain $c \in C_1(X)$ with $\partial c = D$ such that for all $\omega \in \Omega(X)$ we have

$$\int_c \omega = 0,$$

Or, equivalently saying, write D as

$$D = \sum_\nu (z_\nu - w_\nu),$$

Theorem 4.0.2 D has a solution if and only if

$$\varphi(D) := \left(\sum_\nu \int_{w_\nu}^{z_\nu} \omega_1, \cdots, \sum_\nu \int_{w_\nu}^{z_\nu} \omega_g \right) \equiv 0 \mod \Gamma := \mathrm{Per}(\omega_1, \cdots, \omega_g).$$

We have already proved the "if" part, it suffices to prove the "only if" part.

Proof. If D has a solution f, we construct a map $\psi : \mathbb{P}^1 \to \mathrm{Jac}(X)$ as follows:

$$\psi([\lambda_0, \lambda_1]) := \varphi((\lambda_0 f - \lambda_1)).$$

One can check that ψ is well defined and continuous, also, it is holomorphic at points $[1, \lambda]$ if λ is not a branched value of g. Since the other points form a discrete set, ψ is holomorphic. Thus ψ lifts to a holomorphic map $\Psi : \mathbb{P}^1 \to \mathbb{C}^g$, it is constant since each component has to be, thus ψ is constant, it follows that

$$\varphi(D) = \psi([1,0]) = \psi([0,1]) = 0.$$

■